

# Applications of Neutrix Calculus to Special Functions in Conjunction with Polygamma Functions

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**Abstract.** In this paper we define the polygamma functions  $\psi^{(n)}(x)$  for negative integers by using neutrix calculus.

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## 1. Introduction

A rich combination of elementary and special functions arises in the evaluation of Euler sums. Special functions which typically appear are the gamma, beta, digamma and polygamma functions, the zeta functions, polylogarithms, hypergeometric functions, and logarithmic-trigonometric integrals.

Kirchoff was first to apply the polygamma functions in physics, the summation of rational series and the evaluation of integrals are some applications that are still relevant. Recently, the summation of series containing  $\psi^{(n)}(z)$  was arisen in Feynman calculations [2].

Kölbig presented two formulae for the  $\psi^{(n)}(p/q)$  by using the series definition of polylogarithm function, see [16, 17]. Coffey considered the sums over the digamma function, containing summand with  $(\pm 1)^n \psi(n+p/q)/n^2$  and made extension to sums over the polygamma functions in [2].

The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part, see [14].

Using the concepts of the neutrix and the neutrix limit, Fisher gave the general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been exploited in context of distributions and special functions, see [4, 5, 7, 8, 20, 21]

Y. Jack Ng and H. van Dam applied the neutrix calculus, in conjunction with the Hadamard integral, developed by van der Corput see [3], to quantum field theories, in particular, to obtain finite results for the coefficients in the perturbation series. They also applied neutrix calculus to quantum field theory, obtaining finite renormalization in the loop calculations, see [11, 12]

In the following we let  $N$  be the neutrix [3] having domain  $N' = \{\varepsilon : 0 < \varepsilon < \infty\}$  and range  $N''$  the real numbers, with negligible functions finite linear sums of the functions

$$\varepsilon^\lambda \ln^{r-1} \varepsilon, \quad \ln^r \varepsilon \quad (\lambda < 0, \quad r = 1, 2, \dots) \quad (1)$$

and all functions  $f(\varepsilon)$  which converge to zero in the normal sense as  $\varepsilon$  tends to zero.

If  $f(\varepsilon)$  is a real (or complex) valued function defined on  $N'$  and if it is possible to find a constant  $c$  such that  $f(\varepsilon) - c$  is in  $N$ , then  $c$  is called the neutrix limit of  $f(\varepsilon)$  as  $\varepsilon \rightarrow 0$  and we write  $N\text{-}\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = c$ .

Note that if a function  $f(\varepsilon)$  tends to  $c$  in the normal sense as  $\varepsilon$  tends to zero, it converges to  $c$  in the neutrix sense.

Also note that if a function  $H(\varepsilon) = v(\varepsilon) + f(\varepsilon)$ , where  $v(\varepsilon)$  is the sum of negligible functions of  $H(\varepsilon)$ , then p.f.  $H(\varepsilon)$ , Hadamard's finite part of  $H(\varepsilon)$ , is equal to  $f(\varepsilon)$  and so

$$N\text{-}\lim_{\varepsilon \rightarrow 0} H(\varepsilon) = \lim_{\varepsilon \rightarrow 0} f(\varepsilon) = \lim_{\varepsilon \rightarrow 0} p.f.H(\varepsilon).$$

The reader may find the general definition of the neutrix limit with some examples in [3, 4, 5].

In this paper we use Fisher's principle to define the derivative of the digamma function for negative integers. First of all, we give the definition of the gamma function for all  $x$ .

The gamma function  $\Gamma(x)$  is usually defined for  $x > 0$  by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (2)$$

the integral only converging for  $x > 0$ , see [10, 22]. It follows from equation (2) that

$$\Gamma(x+1) = x\Gamma(x) \quad (3)$$

for  $x > 0$  and this equation is used to define  $\Gamma(x)$  for negative, non-integer values of  $x$ . Using the regularization, Gelfand and Shilov [10] define the gamma function

$$\Gamma(x) = \int_0^1 t^{x-1} \left[ e^{-t} - \sum_{i=0}^{n-1} (-1)^i \frac{t^i}{i!} \right] dt + \int_1^\infty t^{x-1} e^{-t} dt + \sum_{i=0}^{n-1} \frac{(-1)^i}{i!(x+i)}$$

for  $x > -n$ ,  $x \neq 0, -1, -2, \dots, -n+1$  and

$$\Gamma(x) = \int_0^\infty t^{x-1} \left[ e^{-t} - \sum_{i=0}^{n-1} (-1)^i \frac{t^i}{i!} \right] dt$$

for  $-n < x < -n+1$ .

Fisher proved that

$$\Gamma(x) = N\text{-}\lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty t^{x-1} e^{-t} dt$$

$x \neq 0, -1, -2, \dots$  and defined  $\Gamma(-m)$  by

$$\begin{aligned}\Gamma(-m) &= \text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} t^{-m-1} e^{-t} dt \\ &= \int_1^{\infty} t^{-m-1} e^{-t} dt + \int_0^1 t^{-m-1} \left[ e^{-t} - \sum_{i=0}^m \frac{(-t)^i}{i!} \right] dt - \sum_{i=0}^{m-1} \frac{(-1)^i}{i!(m-i)}\end{aligned}\quad (4)$$

for  $m = 1, 2, \dots$ , see [6].

More generally, the  $r$ -th derivative of gamma function  $\Gamma(x)$  is defined by

$$\Gamma^{(r)}(x) = \text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} t^{x-1} \ln^r t e^{-t} dt \quad (5)$$

for all  $x$  and  $r = 0, 1, 2, \dots$ , see [8].

Fisher obtained from equation (4) that

$$\Gamma(-m) + \frac{1}{m} \Gamma(-m+1) = \frac{(-1)^m}{mm!}$$

from which it followed by induction that

$$\Gamma(-m) = \frac{(-1)^m}{m!} [\phi(m) - \gamma] \quad (6)$$

for  $m = 1, 2, \dots$ , where  $\gamma$  denotes Euler's constant and

$$\phi(m) = \begin{cases} 0, & m = 0, \\ \sum_{i=1}^m \frac{1}{i}, & m = 1, 2, \dots \end{cases}$$

in particular  $\Gamma(0) = \Gamma'(1) = -\gamma$ , see [6].

The digamma function  $\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$  defined for  $x > 0$  has the integral representation

$$\psi(x) = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt$$

and it can be written as

$$\psi(x) = -\gamma + \int_0^1 \frac{1 - t^{x-1}}{1 - t} dt \quad (x > 0). \quad (7)$$

Differentiating equation (3), we have

$$\Gamma'(x+1) = \Gamma(x) + x\Gamma'(x) \quad (8)$$

for  $x \neq 0, -1, -2, \dots$  and it follows that

$$\psi(x+1) = \psi(x) + \frac{1}{x}$$

see [18, 19] and this equation is used to define  $\psi(x)$  for negative, noninteger values of  $x$ . Thus if  $-m < x < -m + 1$ ;  $m = 1, 2, \dots$ , then

$$\psi(x) = -\gamma + \int_0^1 \frac{1 - t^{x-1+m}}{1 - t} dt - \sum_{k=1}^{m-1} \frac{1}{x + k} \quad (9)$$

In [9] Fisher and kuribayashi proved the following equations to define  $\psi(-m)$  for  $m = 1, 2, \dots$ .

**Theorem 1.1.**

$$\begin{aligned} \Gamma^{(n)}(x) &= \text{N-}\lim_{\varepsilon \rightarrow 0} \Gamma^{(n)}(x + \varepsilon) \\ \Gamma(x + 1) &= \text{N-}\lim_{\varepsilon \rightarrow 0} (x + \varepsilon) \Gamma(x + \varepsilon) \\ \Gamma'(x + 1) &= \text{N-}\lim_{\varepsilon \rightarrow 0} [\Gamma(x + \varepsilon) + (x + \varepsilon) \Gamma'(x + \varepsilon)] \end{aligned}$$

for all  $x$  and  $n = 0, 1, 2, \dots$ .

Theorem 1.1 suggested that the digamma function  $\psi(-m)$  can be defined by

$$\psi(-m) = \text{N-}\lim_{\varepsilon \rightarrow 0} \frac{\Gamma'(-m + \varepsilon)}{\Gamma(-m + \varepsilon)}$$

for  $m = 0, 1, 2, \dots$ , provided the neutrix limit exists, and with this definition

$$\psi(-m) = \psi(1) + \phi(m) = -\gamma + \phi(m) \quad (10)$$

for  $m = 0, 1, 2, \dots$ , see [9].

Recently Tuneska and Jolevski used the integral representation of the digamma function to obtain same result given in [13].

## 2. Defining Polygamma Function $\psi^{(n)}(-m)$

The polygamma function is defined by

$$\psi^{(n)}(x) = \frac{d^n}{dx^n} \psi(x) = \frac{d^{n+1}}{dx^{n+1}} \ln \Gamma(x) \quad (x > 0). \quad (11)$$

It may be represented as

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} dt$$

which holds for  $x > 0$ , and

$$\psi^{(n)}(x) = - \int_0^1 \frac{t^{x-1} \ln^n t}{1 - t} dt. \quad (12)$$

It satisfies the recurrence relation

$$\psi^{(n)}(x+1) = \psi^{(n)}(x) + \frac{(-1)^n n!}{x^{n+1}} \quad (13)$$

see [1, 16, 17, 18, 22]. This is used to define the polygamma function for negative non-integer values of  $x$ . Thus if  $-m < x < -m+1$ ,  $m = 1, 2, \dots$ , then

$$\psi^{(n)}(x) = - \int_0^1 \frac{t^{x+m-1}}{1-t} \ln^n t \, dt - \sum_{k=0}^{m-1} \frac{(-1)^n n!}{(x+k)^{n+1}}. \quad (14)$$

Kölbig gave the formulae for the integral  $\int_0^1 t^{\lambda-1} (1-t)^{-\nu} \ln^m t \, dt$  for integer and half-integer values of  $\lambda$  and  $\nu$  in [15]. As the integral representation of the polygamma function is similar to the integral mentioned above, by using the neutrix limit we prove the existence of the integral in (12) as follows.

Now we let  $N$  be a neutrix having domain the open interval  $\{\epsilon : 0 < \epsilon < \frac{1}{2}\}$  with the same negligible functions as in equation (1). We first of all need the following lemma.

**Lemma 2.1** The neutrix limits as  $\epsilon$  tends to zero of the functions

$$\int_{\epsilon}^{1/2} t^x \ln^n t \ln^r (1-t) \, dt, \quad \int_{1/2}^{1-\epsilon} (1-t)^x \ln^n t \ln^r (1-t) \, dt$$

exists for  $n, r = 0, 1, 2, \dots$  and all  $x$ .

**Proof.** Suppose first of all that  $n = r = 0$ . Then

$$\int_{\epsilon}^{1/2} t^x \, dt = \begin{cases} \frac{2^{-x-1} - \epsilon^{x+1}}{x+1}, & x \neq -1, \\ -\ln 2 - \ln \epsilon, & x = -1 \end{cases}$$

and so  $N\text{-}\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{1/2} t^x \, dt$  exists for all  $x$ .

Now suppose that  $r = 0$  and that  $N\text{-}\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{1/2} t^x \ln^n t \, dt$  exists for some nonnegative integer  $n$  and all  $x$ . Then

$$\int_{\epsilon}^{1/2} t^x \ln^{n+1} t \, dt = \begin{cases} \frac{-2^{-x-1} \ln^{n+1} 2 - \epsilon^{x+1} \ln^{n+1} \epsilon}{x+1} - \frac{n+1}{x+1} \int_{\epsilon}^{1/2} t^x \ln^n t \, dt, & x \neq -1, \\ \frac{(-1)^n \ln^{n+2} 2 - \ln^{n+2} \epsilon}{n+2}, & x = -1 \end{cases}$$

and it follows by induction that  $N\text{-}\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{1/2} t^x \ln^n t \, dt$  exists for  $n = 0, 1, 2, \dots$  and all  $x$ .

Finally we note that we can write

$$\ln^r(1-t) = \sum_{i=1}^{\infty} \alpha_{in} t^i$$

for  $r = 1, 2, \dots$ , the expansion being valid for  $|t| < 1$ . Choosing a positive integer  $k$  such that  $x + k > -1$ , we have

$$\begin{aligned} \int_{\varepsilon}^{1/2} t^x \ln^n t \ln^r(1-t) dt &= \\ &= \sum_{i=1}^{k-1} \alpha_{in} \int_{\varepsilon}^{1/2} t^{x+i} \ln^n t dt + \sum_{i=k}^{\infty} \alpha_{in} \int_{\varepsilon}^{1/2} t^{x+i} \ln^n t dt. \end{aligned}$$

It follows from what we have just proved that

$$\text{N-lim}_{\varepsilon \rightarrow 0} \sum_{i=1}^{k-1} \alpha_{in} \int_{\varepsilon}^{1/2} t^{x+i} \ln^n t dt$$

exists and further

$$\begin{aligned} \text{N-lim}_{\varepsilon \rightarrow 0} \sum_{i=k}^{\infty} \alpha_{in} \int_{\varepsilon}^{1/2} t^{x+i} \ln^n t dt &= \lim_{\varepsilon \rightarrow 0} \sum_{i=k}^{\infty} \alpha_{in} \int_{\varepsilon}^{1/2} t^{x+i} \ln^n t dt \\ &= \sum_{i=k}^{\infty} \alpha_{in} \int_0^{1/2} t^{x+i} \ln^n t dt, \end{aligned}$$

proving that the neutrix limit of  $\int_{\varepsilon}^{1/2} t^x \ln^n t \ln^r(1-t) dt$  exists for  $n, r = 0, 1, 2, \dots$  and all  $x$ . Making the substitution  $1-t = u$  in

$$\int_{1/2}^{1-\varepsilon} (1-t)^x \ln^n t \ln^r t dt,$$

it follows that  $\int_{1/2}^{1-\varepsilon} (1-t)^x \ln^n t \ln^r t dt$  also exists for  $n, r = 0, 1, 2, \dots$  and all  $x$ .

**Lemma 2.2** The neutrix limit as  $\varepsilon \rightarrow 0$  of the integral  $\int_{\varepsilon}^1 t^{-m-1} \ln^n t dt$  exists for  $m, n = 1, 2, \dots$  and

$$\text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 t^{-m-1} \ln^n t dt = -\frac{n!}{m^{n+1}}. \quad (15)$$

**Proof.** Integrating by parts, we have

$$\int_{\varepsilon}^1 t^{-m-1} \ln t dt = m^{-1} \varepsilon^{-m} \ln \varepsilon + m^{-1} \int_{\varepsilon}^1 t^{-m-1} dt$$

and so

$$\text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 t^{-m-1} \ln t dt = -\frac{1}{n^2}$$

proving equation (15) for  $n = 1$  and  $m = 1, 2, \dots$ . Now assume that equation (15) holds for some  $m$  and  $n = 1, 2, \dots$ . Then

$$\begin{aligned} \int_{\varepsilon}^1 t^{-m-2} \ln^n t dt &= (m+1)^{-1} \varepsilon^{-m-1} \ln^n \varepsilon + \frac{n}{m+1} \int_{\varepsilon}^1 t^{-m-2} \ln^{n-1} t dt \\ &= (m+1)^{-1} \varepsilon^{-m-1} \ln^n \varepsilon + \frac{n}{m+1} \frac{-(n-1)!}{(m+1)^n} \end{aligned}$$

and it follows that

$$\text{N-}\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 t^{-m-2} \ln^n t \, dt = -\frac{n!}{(m+1)^{n+1}}$$

proving equation (15) for  $m+1$  and  $n = 1, 2, \dots$ .

Using the regularization and the neutrix limit, we prove the following theorem.

**Theorem 2.3** The function  $\psi^{(n)}(x)$  exists for  $n = 0, 1, 2, \dots$ , and all  $x$ .

**Proof.** Choose positive integer  $r$  such that  $x > -r$ . Then we can write

$$\begin{aligned} \int_{\varepsilon}^{1-\varepsilon} \frac{t^{x-1}}{1-t} \ln^n t \, dt &= \int_{\varepsilon}^{1/2} t^{x-1} \ln^n t \left[ \frac{1}{1-t} - \sum_{i=0}^{r-1} (-1)^i t^i \right] dt \\ &\quad + \sum_{i=0}^{r-1} (-1)^i \int_{\varepsilon}^{1/2} t^{x+i-1} \ln^n t \, dt + \int_{1/2}^{1-\varepsilon} \frac{t^{x-1}}{1-t} \ln^n t \, dt. \end{aligned}$$

We have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{1/2} \frac{t^{x-1}}{1-t} \ln^n t \left[ \frac{1}{1-t} - \sum_{i=0}^{r-1} (-1)^i t^i \right] dt &= \\ &= \int_0^{1/2} \frac{t^{x-1}}{1-t} \ln^n t \left[ \frac{1}{1-t} - \sum_{i=0}^{r-1} (-1)^i t^i \right] dt \end{aligned}$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{1/2}^{1-\varepsilon} \frac{t^{x-1}}{1-t} \ln^n t \, dt = \int_{1/2}^1 \frac{t^{x-1}}{1-t} \ln^n t \, dt$$

the integrals being convergent. Further, from the Lemma 2.1 we see that the neutrix limit of the function

$$\sum_{i=0}^{r-1} (-1)^i \int_{\varepsilon}^{1/2} t^{x+i-1} \ln^n t \, dt$$

exists and implying that

$$\text{N-}\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{1-\varepsilon} \frac{t^{x-1}}{1-t} \ln^n t \, dt$$

exists. This proves the existence of the function  $\psi^{(n)}(x)$  for  $n = 0, 1, 2, \dots$ , and all  $x$ .

Before giving our main theorem, we note that

$$\psi^{(n)}(x) = -\text{N-}\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{t^{x-1}}{1-t} \ln^n t \, dt$$

since the integral is convergent in the neighborhood of the point  $t = 1$ .

**Theorem 2.4** The function  $\psi^{(n)}(-m)$  exists and

$$\psi^{(n)}(-m) = \sum_{i=1}^m \frac{n!}{i^{n+1}} + (-1)^{n+1} n! \zeta(n+1) \quad (16)$$

for  $n = 1, 2, \dots$  and  $m = 0, 1, 2, \dots$ , where  $\zeta(n)$  denotes zeta function.

**Proof.** From Theorem 2.3, we have

$$\begin{aligned}\psi^{(n)}(-m) &= -\text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{t^{-m-1} \ln^n t}{1-t} dt \\ &= -\text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \left[ \sum_{i=1}^{m+1} t^{-i} + (1-t)^{-1} \right] \ln^n t dt.\end{aligned}\quad (17)$$

We first of all evaluate the neutrix limit of integral  $\int_{\varepsilon}^1 t^{-i} \ln^n t dt$  for  $i = 1, 2, \dots$  and  $n = 1, 2, \dots$ .

It follows from Lemma 2.2 that

$$\text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 t^{-i} \ln^n t dt = -\frac{n!}{(i-1)^{n+1}} \quad (i > 1). \quad (18)$$

For  $i = 1$ , we have

$$\int_{\varepsilon}^1 t^{-1} \ln^n t dt = O(\varepsilon).$$

Next

$$\begin{aligned}\int_{\varepsilon}^1 \frac{\ln^n t}{1-t} dt &= \int_0^1 \frac{\ln^n t}{1-t} dt = \sum_{k=0}^{\infty} \int_0^1 t^k \ln^n t dt \\ &= \sum_{k=0}^{\infty} \frac{(-1)^n n!}{(k+1)^{n+1}} \\ &= (-1)^n n! \zeta(n+1),\end{aligned}\quad (19)$$

where

$$\int_0^1 t^k \ln^n t dt = \frac{(-1)^n n!}{(k+1)^{n+1}}.$$

It now follows from equations (17), (18) and (19) that

$$\begin{aligned}\psi^{(n)}(-m) &= -\text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{t^{-m-1} \ln^n t}{1-t} dt \\ &= -\sum_{i=1}^{m+1} \text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 t^{-i} \ln^n t dt - \text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{\ln^n t}{(1-t)} dt. \\ &= \sum_{i=1}^m \frac{n!}{i^{n+1}} + (-1)^{n+1} n! \zeta(n+1)\end{aligned}$$

implying equation (16).

Note that the digamma function  $\psi(x)$  can be defined by

$$\psi(x) = -\gamma + \text{N-lim}_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{1-t^{x-1}}{1-t} dt \quad (20)$$

for all  $x$ .



Using Lemma 2.1 we have

$$\begin{aligned}\int_{\varepsilon}^1 \frac{1-t^{m-1}}{1-t} dt &= -\sum_{i=1}^{m+1} \int_{\varepsilon}^1 t^{-i} dt \\ &= -[\ln 1 - \ln \varepsilon] - \sum_{i=2}^{m+1} \frac{[1 - \varepsilon^{-i+1}]}{-i+1}\end{aligned}$$

and it follows from equation (20) that

$$\begin{aligned}\psi(-m) &= -\gamma + \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{1-t^{m-1}}{1-t} dt = -\gamma + \sum_{i=1}^m i^{-1} \\ &= -\gamma + \phi(m)\end{aligned}$$

which was obtained in [9] and [13].

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